SOME MODIFICATIONS TO LIFTING SURFACE THEORY

by

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1. *Introduction.*

The calculation of pressure distributions in subsonic flow by aid of lifting surface theory is usually performed for a wing, which is symmetrical with respect to the mid-plane of the aircraft [1, 2]. In this case the boundary conditions are clear; at both tips the pressure vanishes while at the mid-plane the pressure either vanishes also (for anti-symmetric loading) or is assumed to have zero derivative in the direction of constant chord fraction (for symmetric loading). For a swept wing this assumption is not conform to reality but its consequences are lessened by an artificial backward shift of the central section, following a proposal by Multhopp [1].

However, there are some configurations where the boundary conditions are less simple, for instance, a horizontal tailplane with endplates or a vertical tailplane which carries the stabilizer with elevator (T-tail or crosstail). Considering in particular the last configuration, the boundary condition for the vertical tailplane at the position of the horizontal tailplane gives some difficulty since the latter is no plane of symmetry for the vertical tailplane and hence, the derivative of the pressure in the direction of constant fraction of the vertical tailplane chord will not vanish there.

This problem has been considered by Davies [3], who has given a solution which, however, carries some elements of arbitrariness in it. By using the theory of orthogonal functions the present author comes to a solution, which seems to be more logical and which, moreover, can be obtained for a greater part by analytical instead of numerical calculations. This procedure has also certain advantages when calculating the pressure distribution due to a symmetrical loading for a swept wing near its central section. In that case the assumption of zero derivative of the pressure in the relevant direction does not need to be introduced and a potential-theoretical solution, which is more accurate near the mid-section, can be obtained with the present method.

By way of example the procedure has been applied to a swept wing, the same which has been considered by Multhopp [1]. There appeared some difficulties with regard to the convergence of the numerical solution towards the exact solution if the number of pivotal points in chordwise direction is increased. The difficulties are not specific for the present procedure since they exist at least as seriously in the work of Lashka [2]. Their elimination can be performed even more easily in the present method, thus leading to another advantage for this method.

An Algol 60 program for symmetrical loading of a swept wing in incompressible flow is available.

2. *The choice of the system of orthogonal functions.*

The problem to be considered is the spanwise integration occurring in lifting surface theory which is concerned with integrals of a. o. the following type

$$\oint_0^1 \frac{f(\eta')}{(\eta - \eta')^2}\, d\eta' . \tag{1}$$

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This integral must be taken as

$$\lim_{\epsilon \to 0} \left\{ \int_0^{\eta-\epsilon} \frac{f(\eta')}{(\eta-\eta')^2}\, d\eta' + \int_{\eta+\epsilon}^{1} \frac{f(\eta')}{(\eta-\eta')^2}\, d\eta' - 2\,\frac{f(\eta)}{\epsilon} \right\},$$

which means that the function $(\eta-\eta')^{-2}$ should be considered as a distribution or generalized function [4].

The function $f(\eta)$ is obtained from a chordwise integration containing the pressure distribution.

Following Multhopp [1], the integration in (2.1) is performed by first introducing an interpolation formula

$$f(\theta) = \sum_{n=1}^{m+1} f(\theta_n)\, g_n(\theta), \tag{2.2}$$

where θ is some angular coordinate ranging from $\theta = \frac{\pi}{2}$ at the root ($\eta = 0$) to $\theta = 0$ at the tip ($\eta = 1$). Multhopp takes $\eta = \cos\theta$, but we shall obtain a different relation. The functions $g_n(\theta)$ are so-called station functions which are zero at all stations $\theta = \theta_\nu$, $\nu = 1, 2, \ldots, m+1$ except at $\theta = \theta_n$ where $g_n(\theta)$ is equal to 1. This shows immediately the correctness of (2.2) at the station points.

Hence

$$g_n(\theta_\nu) = \delta_{n\nu}. \tag{2.3}$$

The station functions will now be expanded into a system of orthogonal functions, which is also analogous to Multhopp's procedure. The problem, however, is the choice of the orthogonal functions. At the interval $0 \leqslant \theta \leqslant \frac{\pi}{2}$ there are four different complete systems of Fourier expansion possible, viz. towards the functions

$$\cos(2\lambda+1)\theta, \quad \cos 2\lambda\theta, \quad \sin(2\lambda+1)\theta \quad \text{and} \quad \sin 2\lambda\theta,$$

where $\lambda = 0, 1, \ldots$.

Which of these expansions is preferable in a particular case depends upon the boundary conditions for the function $f(\theta)$, since this function in its turn must be expanded into the system of station functions, according to (2.2). At the tip $\theta = 0$ the function $f(\theta)$ should vanish and this eliminates the systems $\cos(2\lambda+1)\theta$ and $\cos 2\lambda\theta$. At the root, $\theta = \pi/2$, $f(\theta)$ should not vanish and this makes the system $\sin 2\lambda\theta$ highly impractical. Therefore, the best choice in our case is the system

$$\sin(2\lambda+1)\theta, \quad \lambda = 0, 1, \ldots$$

The derivative to θ at the root $\theta = \frac{\pi}{2}$ then vanishes, but due to the relation which we shall derive between η and θ, it appears that the derivative to η at the root will be different from zero.

The orthogonality and normalizing relation for our system is

$$\frac{4}{\pi} \int_0^{\pi/2} \sin(2\lambda+1)\theta \, \sin(2\mu+1)\theta \, d\theta = \delta_{\lambda\mu}. \tag{2.4}$$

Such an orthogonality relation in the form of an integral can always be transformed by aid of the theory of orthogonal polynomials [5] into an orthogonality relation in the form of a summation. This then leads in a natural way to both the position of the stations θ_n and the coefficients $a_{\lambda,n}$ in the expansions

$$g_n(\theta) = \sum_{\lambda=0}^{\infty} a_{\lambda,n} \sin(2\lambda+1)\theta .$$

The procedure to transform (2.4) into a summation orthogonality relation is as follows.

By putting $\cos\theta = x$, we have

$$\sin(2\lambda+1)\theta = \sqrt{1-x^2}\; P_\lambda(x^2),$$

where $P_\lambda(x^2)$ is a polynomial in x^2 of degree λ. It may be noted that

$$P_\lambda(x^2) = \frac{\sin(2\lambda+1)\theta}{\sin\theta} = U_{2\lambda}(x),$$

where $U_{2\lambda}(x)$ are the Chebyshev polynomials of the second kind.

The orthogonality relation becomes

$$\frac{4}{\pi}\int_0^1 \sqrt{1-x^2}\; P_\lambda(x^2)\, P_\mu(x^2)\,dx = \delta_{\lambda\mu} .$$

In this formula only x^2 occurs but not x itself. Therefore it is more logical to introduce

$$\eta = x^2 = \cos^2\theta,$$

which brings the orthogonality relation in the form

$$\frac{2}{\pi}\int_0^1 \sqrt{\frac{1-\eta}{\eta}}\; P_\lambda(\eta)\, P_\mu(\eta)\,d\eta = \delta_{\lambda\mu}, \tag{2.5}$$

where now $P_\lambda(\eta)$ is a polynomial in η of degree λ.

From the relation

$$\sin(2\lambda+3)\theta + \sin(2\lambda-1)\theta = 2\sin(2\lambda+1)\theta\cos 2\theta$$

immediately follows the three-term recurrence relation between the polynomials $P_\lambda(\eta)$,

$$P_{\lambda+1}(\eta) + P_{\lambda-1}(\eta) = 2(2\lambda-1)P_\lambda(\eta).$$

From the existence of such a recurrence relation follows that the polynomials $P_\lambda(\eta)$ also satisfy the summation formula of Christoffel-Darboux [5, p. 126]

$$\sum_{\lambda=0}^{m} P_\lambda(\eta_n)P_\lambda(\eta_\nu) = \frac{1}{4}\,\frac{P_{m+1}(\eta_n)P_m(\eta_\nu) - P_m(\eta_n)P_{m+1}(\eta_\nu)}{\eta_n - \eta_\nu}, \tag{2.6}$$

where η_n and η_ν are arbitrary values of η. If, however, we take for η_n and η_ν two different zeroes of $P_{m+1}(\eta)$ we find

$$\sum_{\lambda=0}^{m} P_\lambda(\eta_n)P_\lambda(\eta_\nu) = 0, \tag{2.7}$$

where $P_{m+1}(\eta_n) = P_{m+1}(\eta_\nu) = 0, \quad \eta_n \neq \eta_\nu$.
Since

$$P_{m+1}(\eta) = \frac{\sin(2m+3)\theta}{\sin\theta}; \qquad \eta = \cos^2\theta \tag{2.8}$$

the zeroes of $P_{m+1}(\eta)$ are given by

$$\theta_n = \frac{n}{2m+3}\pi \qquad n = 1, 2, \ldots, m+1, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

If in (2.6) η_ν approaches η_n, it follows that in the limit

$$\sum_{\lambda=0}^{m} P_\lambda^2(\eta_n) = \tfrac{1}{4}\{P'_{m+1}(\eta_n)P_m(\eta_n) - P'_m(\eta_n)P_{m+1}(\eta_n)\}.$$

Taking again for η_n a zero of $P_{m+1}(\eta)$, the result is

$$\sum_{\lambda=0}^{m} P_\lambda^2(\eta_n) = \tfrac{1}{4} P'_{m+1}(\eta_n)P_m(\eta_n).$$

Using (2.8) it is found after some reductions that

$$\sum_{\lambda=0}^{m} P_\lambda^2(\eta_n) = \frac{2m+3}{4\sin^2\theta_n}$$

or

$$\sum_{\lambda=0}^{m} \sin^2(2\lambda+1)\theta_n = \frac{2m+3}{4}.$$

Combining with (2.7), we obtain the orthogonality relation in summation form

$$\frac{4}{2m+3}\sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \sin(2\lambda+1)\theta_\nu = \delta_{n\nu}. \tag{2.9}$$

Comparing this with eq. (2.3), it is clear that we may write

$$g_n(\theta) = \frac{4}{2m+3}\sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \sin(2\lambda+1)\theta. \tag{2.10}$$

When we would perform this same procedure for the system $\sin 2\lambda\theta$ (corresponding to vanishing pressure at both ends) we would obtain the same result for $g_n(\theta)$ as in [1] although Multhopp came to it in a more intuitive way.

We identify η with the spanwise coordinate η used in (2.1). The stations η_n are given by

$$\eta_n = \cos^2\theta_n, \qquad \theta_n = \frac{n}{2m+3}\pi, \qquad n = 1, 2, \ldots, m+1 \tag{2.11}$$

It is seen from table 1 that the density of the stations is largest near the root and near the tip. This agrees with the general requirement in approximation theory, that one always needs a larger density of points near the ends of the interval.

TABLE 1. Position of stations for m=7

n	1	2	3	4	5	6	7	8
η_n	0.9662	0.8695	0.7229	0.5461	0.3632	0.1987	0.0749	0.0085

It may be remarked that Davies [3] uses as orthogonality relation instead of (2.5)

$$\int_0^1 \sqrt{1-\eta}\; P_\lambda(\eta) P_\mu(\eta) d\eta = \delta_{\lambda\mu},$$

which is satisfied by certain hypergeometric functions. Davies then does not write the station functions $g_n(\eta)$ as series of hypergeometric function, but he derives the relation

$$g_n(\eta) = \frac{P_{m+1}(\eta)}{(\eta-\eta_n)\left\{\frac{d}{d\eta} P_{m+1}(\eta)\right\}_{\eta=\eta_n}} \sqrt{\frac{1-\eta}{1-\eta_n}}, \tag{2.12}$$

where η_n are the zeroes of $P_{m+1}(\eta)$.

In our case the same relation (2.12) holds but with $\eta = \cos^2\theta$, η_n given by eq.(2.11) and $P_{m+1}(\eta)$ given by eq.(2.8). This last result may be derived by performing the summation in (2.10) by aid of (2.6), which leads to

$$g_n(\theta) = \frac{\sin(2m+1)\theta_n}{\cos^2\theta - \cos^2\theta_n} \; \frac{\sin(2m+3)\theta}{2m+3}.$$

and which can be identified with eq.(2.12).

3. *The quadrature formulae for the spanwise integration.*

A number of spanwise integrals, of which (2.1) gives an example, occur and these will be evaluated by aid of the interpolation formula (2.2). The results then are obtained in the form of linear combinations of the ordinates $f(\eta_n)$. The coefficients depend upon the station η_ν for which most of the integrals have to be evaluated. These stations are the same stations as used for the interpolation. They are defined by eq.(2.11). Then

$$\left.\begin{aligned} \oint_0^1 \frac{f(\eta')}{(\eta_\nu-\eta')^2} d\eta' &= -\sum_{n=1}^{m+1} b_{\nu n} f(\eta_n) \\ \oint_0^1 \frac{f(\eta')}{\eta_\nu-\eta'} d\eta' &= \sum_{n=1}^{m+1} e_{\nu n} f(\eta_n) \\ \int_0^1 f(\eta')\ln|\eta_\nu-\eta'|\, d\eta' &= \sum_{n=1}^{m+1} c_{\nu n} f(\eta_n) \end{aligned}\right\} \nu = 1,2,\dots,m+1 \tag{3.1}$$

$$\int_0^1 f(\eta')d\eta' = \sum_{n=1}^{m+1} d_n f(\eta_n). \tag{3.2}$$

There exists also an interference with the surface at the other side of the root section $\eta = 0$. It will now be assumed that the "semi-spans" of both sides of the lifting surface are equal and that $\eta = 0$ is a symmetry plane. It would be only a matter of simple arithmetics to extend the following theory to the case of different "semi-spans", but it is thought that this makes no sense if the vertical plane at the root section $\eta = 0$ is not at the same time introduced as boundary condition. This, however, would involve the whole interference problem of a cross-tail with formulae which are much more complicated but give no deeper insight. Therefore, we shall restrict ourselves in the following to the investigations of a swept-wing for which the present procedure has also advantages. The left wing gives rise to the following integrals

$$\left.\begin{aligned} \int_{-1}^{0} \frac{f(\eta')}{(\eta_\nu - \eta')^2}\, d\eta' &= -\sum_{n=m+2}^{2m+2} b_{\nu n} f(\eta_n) \\ \int_{-1}^{0} \frac{f(\eta')}{\eta_\nu - \eta'}\, d\eta' &= \sum_{n=m+2}^{2m+2} e_{\nu n} f(\eta_n) \\ \int_{-1}^{0} f(\eta') \ln|\eta_\nu - \eta'|\, d\eta' &= \sum_{n=m+2}^{2m+2} c_{\nu n} f(\eta_n) \end{aligned}\right\} \quad \nu = 1, 2, \ldots, m+1\;^{1)} \qquad (3.3)$$

The sections η_n with n from m+2 to 2m+2 are given by

$$\eta_{n+m+1} = -\cos^2\theta_n, \quad \theta_n = \frac{n}{2m+3}\pi, \quad n = 1, 2, \ldots, m+1. \qquad (3.4)$$

For $\nu = m+2, \ldots, 2m+2$ the formulae are

$$\int_{0}^{1} \frac{f(\eta')}{(\eta_\nu - \eta')^2}\, d\eta' = -\sum_{n=1}^{m+1} b_{\nu-m-1,\, n+m+1} f(\eta_n),$$

$$\oint_{-1}^{0} \frac{f(\eta')}{(\eta_\nu - \eta')^2}\, d\eta' = -\sum_{n=m+2}^{2m+2} b_{\nu-m-1,\, n-m-1} f(\eta_n)$$

with obvious results for the other formulae.

The minus sign in the formulae with the coefficients $b_{\nu n}$ has been added in order to keep the same definition for these coefficients as in [2]. This notation differs from that in [1] in the sign of $b_{\nu n}$ if $\nu \neq n$; $\nu, n = 1, 2, \ldots, m+1$.

By substitution of eqs. (2.2), (2.10) and (2.11) in the integrals, it is found that for $\nu, n = 1, 2, \ldots, m+1$

$$\left.\begin{aligned} b_{\nu n} &= -\frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \oint_0^{\pi/2} \frac{\sin(2\lambda+1)\theta' \sin 2\theta'}{(\cos^2\theta_\nu - \cos^2\theta')^2}\, d\theta' \\ e_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \oint_0^{\pi/2} \frac{\sin(2\lambda+1)\theta' \sin 2\theta'}{\cos^2\theta_\nu - \cos^2\theta'}\, d\theta' \\ c_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \int_0^{\pi/2} \sin(2\lambda+1)\theta' \sin 2\theta' \ln|\cos^2\theta_\nu - \cos^2\theta'|\, d\theta' \\ d_n &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \int_0^{\pi/2} \sin(2\lambda+1)\theta' \sin 2\theta'\, d\theta'. \end{aligned}\right\} \qquad (3.5)$$

1) In the case of different "semi-spans", $\eta_\nu - \eta'$ would have to be replaced by $\eta_\nu - \zeta\eta'$, where ζ denotes the ratio of the semi-span of the left to that of the right surface.

Expressions for $b_{\nu n}$, $e_{\nu n}$ and $c_{\nu n}$ if $n = m+2, \dots, 2m+2$ are obtained by replacement of $\cos^2\theta_\nu - \cos^2\theta'$ by $\cos^2\theta_\nu + \cos^2\theta'$.

Further reduction of the expressions (3.5) leads to

$$\left.\begin{aligned}
b_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ \frac{(-1)^\lambda}{\cos^2\theta_\nu} - (2\lambda+1) I_\lambda \right\} \\
e_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ I_{\lambda-1} - I_{\lambda+1} \right\} \\
c_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ \frac{(-1)^\lambda}{2} \left(\frac{1}{2\lambda+3} - \frac{1}{2\lambda-1} \right) \ln \cos^2\theta_\nu + \right. \\
&\qquad \left. + \frac{1}{4(2\lambda-1)} (I_\lambda - I_{\lambda-2}) + \frac{1}{4(2\lambda+3)} (I_\lambda - I_{\lambda+2}) \right\} \\
d_n &= \frac{2}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ (-1)^\lambda \left(\frac{1}{2\lambda+3} - \frac{1}{2\lambda-1} \right) \right\}.
\end{aligned}\right\} \quad (3.6)$$

For $n = m+2, \dots, 2m+2$ the results are

$$\left.\begin{aligned}
b_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ - \frac{(-1)^\lambda}{\cos^2\theta_\nu} + (2\lambda+1) J_\lambda \right\} \\
e_{\nu n} &= \frac{2}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ J_{\lambda-1} - J_{\lambda+1} \right\} \\
c_{\nu n} &= \frac{4}{2m+3} \sum_{\lambda=0}^{m} \sin(2\lambda+1)\theta_n \left\{ \frac{(-1)^\lambda}{2} \left(\frac{1}{2\lambda+3} - \frac{1}{2\lambda-1} \right) \ln \cos^2\theta_\nu - \right. \\
&\qquad \left. - \frac{1}{4(2\lambda-1)} (J_\lambda - J_{\lambda-2}) - \frac{1}{4(2\lambda+3)} (J_\lambda - J_{\lambda+2}) \right\}.
\end{aligned}\right\} \quad (3.7)$$

while the formula for d_n is the same as for $n = 1, 2, \dots, m+1$.

The formulae for I_λ and J_λ are as follows

$$I_\lambda = \oint_0^{\pi/2} \frac{\cos(2\lambda+1)\theta'}{\cos^2\theta_\nu - \cos^2\theta'} \, d\theta', \quad J_\lambda = \int_0^{\pi/2} \frac{\cos(2\lambda+1)\theta'}{\cos^2\theta_\nu + \cos^2\theta'} \, d\theta'. \quad (3.8)$$

The integral I_λ, which can also be written as

$$I_\lambda = \oint_0^{\pi} \frac{\cos(\lambda+\frac{1}{2})\psi'}{\cos\psi_\nu - \cos\psi'} \, d\psi', \quad \psi_\nu = 2\theta_\nu,$$

differs from the well-known Glauert's integral [6, p. 173] by the replacement of λ by $\lambda + \frac{1}{2}$ with λ integer. The integrals I_0 and J_0 are evaluated by putting

$$\tan\theta_\nu = z_\nu \quad \text{and} \quad \tan\theta' = z'$$

with the result that

$$\left.\begin{aligned}
&I_0 = \frac{1}{\sin\theta_\nu} \ln \frac{1 - \sin\theta_\nu}{\cos\theta_\nu}, \quad J_0 = \frac{1}{\operatorname{Cosh}\xi_\nu} \ln \frac{1 + \operatorname{Cosh}\xi_\nu}{\operatorname{Sinh}\xi_\nu}, \\
&\qquad \text{where } \operatorname{Sinh}\xi_\nu = \cos\theta_\nu.
\end{aligned}\right\} \quad (3.9)$$

Since $I_{-1} = I_0$ and $J_{-1} = J_0$, the other I_λ and J_λ are determined by the recurrence relations (λ integer)

$$\left.\begin{aligned} I_{\lambda+1} + I_{\lambda-1} &= -\frac{4}{2\lambda+1}(-1)^\lambda + 2I_\lambda \cos 2\theta_\nu , \\ J_{\lambda+1} + J_{\lambda-1} &= \frac{4}{2\lambda+1}(-1)^\lambda - 2J_\lambda \operatorname{Cosh} 2\xi_\nu . \end{aligned}\right\} \qquad (3.10)$$

It can be shown from the characteristic equation, that the recurrence relation for I_λ is stable, but that the relation for J_λ is unstable. How the latter difficulty is to be circumvented, will be dealt with in Sec. 5.

4. *The integral equation for the pressure difference.*

According to ref. [1] and [2], the integral equation for the dimensionless pressure difference

$$\Delta c_p = \frac{p_l - p_u}{\frac{1}{2}\rho U^2}$$

is in incompressible flow

$$\alpha(x,y) = -\frac{1}{8\pi} \oint_{-s}^{s} \int_{x_l(y')}^{x_t(y')} \frac{\Delta c_p(x',y')}{(y-y')^2} \left\{1 + \frac{x-x'}{\sqrt{(x-x')^2+(y-y')^2}}\right\} dx'dy'. \qquad (4.1)$$

x and x' are coordinates in chordwise direction, y and y' in spanwise direction, s is the semi-span, $x_l(y')$ denotes the x-coordinate of the leading edge, $x_t(y')$ that of the trailing edge, p_l is the pressure at the lower side, p_u that at the upper side, ρ is the air density and U the speed.

The following symbols will also be used

$$\left.\begin{aligned} &l(y') = x_t(y') - x_l(y'), \quad X' = \frac{x' - x_l(y')}{l(y')}, \\ &X = \frac{x - x_l(y')}{l(y')}, \quad X_a = \frac{x - x_l(y)}{l(y)}, \quad Y = \frac{y-y'}{l(y)} . \end{aligned}\right\} \qquad (4.2)$$

The quantities X' and X_a vary between 0 and 1. This does not hold for X.

For the pressure difference in chordwise direction we assume a Birnbaum series with coefficients depending on the spanwise coordinate. Hence

$$\left.\begin{aligned} &\Delta c_p(x',y') = \frac{4s}{l(\eta')} \sum_{r=0}^{R} a_r(\eta')\, h_r(X'), \\ &\text{where } h_0(X') = \cot\frac{\varphi}{2}, \quad h_r(X') = \sin r\varphi \quad \text{if} \quad r \geqslant 1, \\ &X' = \frac{1-\cos\varphi}{2}, \quad \eta' = y'/s, \quad l(\eta') = l(y') . \end{aligned}\right\} \qquad (4.3)$$

After some reductions which are analogous to those performed in ref. [2], and which are also described in more detail in ref. [7], the integral equation can be written in the form

$$\left.\begin{aligned} \alpha(x,y) &= -\frac{1}{2\pi}\sum_{r=0}^{R}\left\{\oint_{-1}^{1}\frac{a_r(\eta')f_{1r}(\xi,\eta,\eta')}{(\eta-\eta')^2}d\eta' + f_{2r}(\xi,\eta)\int_{-1}^{1}a_r(\eta')\ln|\eta-\eta'|\,d\eta'\right\}, \\ \text{where } f_{1r}(\xi,\eta,\eta') &= \int_{0}^{1}h_r(X')\left\{1+\frac{X-X'}{\sqrt{(X-X')^2+Y^2}}\right\}dX' + \\ &\quad + \frac{s^2}{\{l(\eta)\}^2}\left(\frac{dh_r}{dX}\right)_{X=X_a}(\eta-\eta')^2\ln|\eta-\eta'|, \\ f_{2r}(\xi,\eta) &= -\frac{s^2}{\{l(\eta)\}^2}\left(\frac{dh_r}{dX}\right)_{X=X_a}, \quad \xi = X(y'), \quad \eta = y/s. \end{aligned}\right\} \quad (4.4)$$

For the integration over the left surface ($-1 < \eta' < 0$), $l(\eta)$ has to be replaced by $\bar{l}(\eta)$ in the formulae for $f_{1r}(\xi,\eta,\eta')$ and $f_{2r}(\xi,\eta)$, where $\bar{l}(\eta)$ denotes the imaginary chord obtained at the section η when the left wing is analytically continued up to this section (see ref. [7] for details).

5. *Solution of the integral equation.*

As we have restricted the freedom of $\Delta c_p(x',y')$ by approximating it by the series of R+1 terms given in eq. (4.3), it is only possible to satisfy eq. (4.4) for the R+1 values of x, defined by

$$x_\mu = x_1 + \frac{1}{2}\left(1-\cos\frac{2\pi\mu}{2R+3}\right), \quad \mu = 1,2,\ldots,R+1. \tag{5.1}$$

Similarly, since the functions $a_r(\eta')$ will be approximated by interpolation formulae of the type given by (2.2), corresponding to a Fourier series of m+1 terms according to (2.10), the integral equation (4.4) can be satisfied only for m+1 values of y at each semi-wing. These values of y are given by eq. (2.11) for the right semi-wing and by eq. (3.4) for the left semi-wing.

In this way there have been defined (m+1)(R+1) pivotal points (x_μ, y_ν) at which eq. (4.4) will be satisfied. The integral equation is then replaced by the set of algebraic equations

$$\left.\begin{aligned} \alpha(x_\mu,y_\nu) &= \frac{1}{2\pi}\sum_{r=0}^{R}\sum_{n=1}^{2m+2}\{b_{\nu n}f_{1r}(\xi_\mu,\eta_\nu,\eta_n) - c_{\nu n}f_{2r}(\xi_\mu,\eta_\nu)\}\,a_r(\eta) \\ \mu &= 1,2,\ldots,R+1\,; \quad \nu = 1,2,\ldots,m+1. \end{aligned}\right\} \quad (5.2)$$

The quantities $f_{1r}(\xi_\mu,\eta_\nu,\eta_n)$ are evaluated for $r \geqslant 1$ after integration by parts, see [7].

In the calculation of the coefficients $b_{\nu n}$ and $c_{\nu n}$ for n=m+2, ..., 2m+2 there is a complication due to the instability in the evaluation by recurrence of the quantities J_λ. Since the characteristic equation of the second eq. (3.10) is

$$t^2 + 2t\,\mathrm{Cosh}\,2\xi_\nu + 1 = 0,$$

it follows that the dominant term in the error increases as $(-1)^p e^{2\xi_\nu p}$, where p is the number of times the recurrence relation has been applied. Since the accuracy of the Telefunken TR4, on which the calculations have been performed, is 11 digits in floating form, a loss of 7 digits was considered to be the limit which could be accepted. Hence, the recurrence relation is only used if

$$e^{2\xi_\nu (m+1)} < 10^7 \tag{5.3}$$

and then $b_{\nu n}$ and $c_{\nu n}$ follow from eq.(3.7). Eq.(5.3) will not be satisfied if either ξ_ν or m is large enough. ξ_ν large implies θ_ν small and η_ν far from the section $\eta = 0$, which means that the singularity at $\eta' = \eta_\nu$ is far outside the interval of integration. m large means that a large number of points is used for the spanwise integration. In both cases it is admitted to neglect the singularity outside the interval of integration and to use eq.(3.2) for the integration. Then

$$b_{\nu n} = -\frac{d_n}{(\eta_\nu - \eta_n)^2}, \quad c_{\nu n} = d_n \ln|\eta_\nu - \eta_n|, \quad n=m+2,\ldots,2m+2$$

where d_n is given by eq.(3.6).

Eq.(5.2) has been solved for $a_r(\eta_n)$ and from these results the sectional lift and moment as well as the total lift and centre of pressure were calculated with formulae given in [7]. However, it appeared that convergence with increasing m was rather slow, especially if R was 2 or more. This is due to the inaccuracy of the spanwise integration. The results for the total lift of a flat wing with $\alpha = 1$ and of the resultant center of pressure of that wing are given in table 2 for various m and R. The wing is the same as that considered by Multhopp [1, page 67].

TABLE 2. Results obtained from eq.(5.2)

Slope of life curve, $\frac{\partial c_L}{\partial \alpha}$.

R+1 ↓ \ m+1 →	8	12	16	20	24
1	3.2589	3.2639	3.2647	3.2650	3.2652
2	3.3272	3.2967	3.2914	3.2926	3.2940
3	3.3423	3.3318	3.3137	3.3020	3.2965
4	3.3094	3.3242	3.3242	3.3182	

Position of the resultant centre of pressure given as fraction of the root chord.

R+1 ↓ \ m+1 →	8	12	16	20	24
2	0.8267	0.8305	0.8301	0.8294	0.8290
3	0.8176	0.8239	0.8278	0.8295	0.8299
4	0.8184	0.8204	0.8230	0.8254	

The slow convergence is not specific for the integration rules used in this paper, since they have also been encountered at the NLR in Amsterdam by Zandbergen and his group, using the Multhopp distribution of pivotal points in spanwise direction. They also occur for non-swept wings including the rectangular wing.

In order to overcome these difficulties it has been proposed by Zandbergen [8] to expand $f_{1r}(\xi, \eta, \eta')$ near $\eta' = \eta$ in the Taylor series

$$f_{1r}(\xi,\eta,\eta') = f_{1r}(\xi,\eta,\eta) + (\eta'-\eta)\frac{\partial f_{1r}}{\partial n'}(\xi,\eta,\eta) + R_{1r}(\xi,\eta,\eta'), \tag{5.4}$$

where $R_{1r}(\xi,\eta,\eta')$ contains a factor $(\eta'-\eta)^2$. The integral equation then assumes the form

$$\alpha(\xi,\eta) = -\frac{1}{2\pi}\sum_{r=0}^{R}\left\{f_{1r}(\xi,\eta,\eta)\oint_{-1}^{1}\frac{a_r(\eta')}{(\eta-\eta')^2}d\eta' + \left(\frac{\partial f_{1r}(\xi,\eta,\eta')}{\partial\eta'}\right)_{\eta'=\eta}\oint_{-1}^{1}\frac{a_r(\eta')}{\eta'-\eta}d\eta'\right.$$
$$\left.+\int_{-1}^{1}\frac{a_r(\eta')R_{1r}(\xi,\eta,\eta')}{(\eta-\eta')^2}d\eta' + f_{2r}(\xi,\eta)\int_{-1}^{1}a_r(\eta')\ln\left|\eta-\eta'\right|d\eta'\right\}. \tag{5.5}$$

Accepting the expansion of $a_r(\eta')$ in a Fourier series of m+1 terms, the integrals

$$\oint_{-1}^{1}\frac{a_r(\eta')}{(\eta-\eta')^2}d\eta', \quad \oint_{-1}^{1}\frac{a_r(\eta')}{\eta'-\eta}d\eta' \quad \text{and} \quad \int_{-1}^{1}a_r(\eta')\ln|\eta-\eta'|d\eta'$$

can be calculated exactly. The remaining integral has an integrand which is finite for $\eta'=\eta$ unless $\eta=0$. Zandbergen suggested to calculate this integral by aid of an increased number of points. In the present investigation this integral has been calculated by aid of eq.(3.2). It is thought that the error thus made in the evaluation of the integral is of the same order as that made by restricting the Fourier series of $a_r(\eta')$ to m+1 terms.

The main advantage of eq.(5.5) over eq.(4.4) can be explained as follows. The accuracy of the first formula of eq.(3.1) is not too good (unless $f(\eta')$ consists of a m+1 term Fourier series, in which case it is exact), since $b_{\nu\nu}$ is rather large positive while $b_{\nu,\nu-1}$ and $b_{\nu,\nu+1}$ are negative in such a way that these three terms cancel each other for an important part. Therefore, the factor $f_{1r}(\xi,\eta,\eta')$ occurring in eq.(4.4) after the integralsign gives errors, additional to and dominating those of the corresponding term in eq.(5.5).

The set of algebraic equations used is

$$\alpha(\xi_\mu,\eta_\nu) = \frac{1}{2\pi}\sum_{r=0}^{R}\sum_{n=1}^{2m+2}\left\{b_{\nu n}f_{1r}(\xi_\mu,\eta_\nu,\eta_\nu) + e_{\nu n}\left(\frac{\partial f_{1r}(\xi_\mu,\eta_\nu,\eta')}{\partial\eta'}\right)_{\eta'=\eta_\nu} -\right.$$
$$\left.- d_n\frac{R_{1r}(\xi_\mu,\eta_\nu,\eta_n)}{(\eta_\nu-\eta_n)^2} - c_{\nu n}f_{2r}(\xi_\mu,\eta_\nu)\right\}a_r(\eta_n). \tag{5.6}$$

The coefficients $b_{\nu n}$, $c_{\nu n}$, $e_{\nu n}$ and d_n are obtained from eqs.(3.6) and (3.7), but (3.7) again with the restriction that if (5.3) is not satisfied, we have instead

$$b_{\nu n} = -\frac{d_n}{(\eta_\nu-\eta_n)^2}, \quad e_{\nu n} = \frac{d_n}{\eta_\nu-\eta_n}, \quad c_{\nu n} = d_n\ln|\eta_\nu-\eta_n|, \quad n=m+2,\ldots,2m+2.$$

The results obtained from eq.(5.6) for the total lift and centre of pressure of the same wing as considered before, are given in table 3. By comparison with table 2 it is seen that the convergence with increasing m is much better when using eq.(5.6).

TABLE 3. Final, more accurate, results obtained from eq.(5.6)

Slope of lift curve, $\frac{\partial c_L}{\partial \alpha}$.

R+1↓ \ m+1→	8	12	16	20	24
1	3.2661	3.2658	3.2657	3.2656	3.2656
2	3.3000	3.2967	3.2958	3.2957	3.2956
3	3.2856	3.2977	3.2987	3.2977	3.2969
4	3.2689	3.2863	3.2946	3.2977	

Position of the resultant centre of pressure given as fraction of the root chord.

R+1↓ \ m+1→	8	12	16	20	24
2	0.8273	0.8284	0.8286	0.8286	0.8286
3	0.8277	0.8271	0.8277	0.8282	0.8285
4	0.8310	0.8283	0.8275	0.8275	

Results obtained by Multhopp [1], 15 stations on total span

1 chordwise station $\partial c_L/\partial\alpha = 3.232$

2 chordwise stations $\partial c_L/\partial\alpha = 3.275$, while the centre of pressure is at 0.829 of the root chord.

Another property of the result is that increase of R always should be accompanied by increase of m if an improvement of the results is aimed at. The expansion of $a_r(\eta')$ in a series of m+1 terms becomes apparently more difficult for increasing r. This can be made plausible for a rectangular wing where the higher terms in the Birnbaum series are more concentrated near the tips.

For the details of the calculations the reader is referred to [7].

Finally, it may be mentioned that the evaluation of

$$\frac{R_{1r}(\xi_\mu, \eta_\nu, \eta_n)}{(\eta_\nu - \eta_n)^2}$$

for $\eta_n = \eta_\nu$, occurring in eq.(5.6), is only possible if there are no kinks in the leading or trailing edge at any of the spanwise sections η_n. Since in the present procedure all integrations are performed separately over both semi-wings, the middle section where such a kink often occurs, is not one of the sections η_n. This is another advantage over the classical methods of [1] and [2], where the introduction of the Taylor series (5.4) gives complications.

6. *Results.*

Numerical computations have been performed for a number of flat wings. We present here the results for a wing, which has also been investigated by Multhopp [1]. It is shown in fig.1. Leading and trailing edges are straight for each semi-wing. The angle of sweep of the leading edge is 45°. The aspect ratio is 4. The wing is tapered with a root chord l(0) = 0.7 s and a tip chord l(1) = 0.3 s.

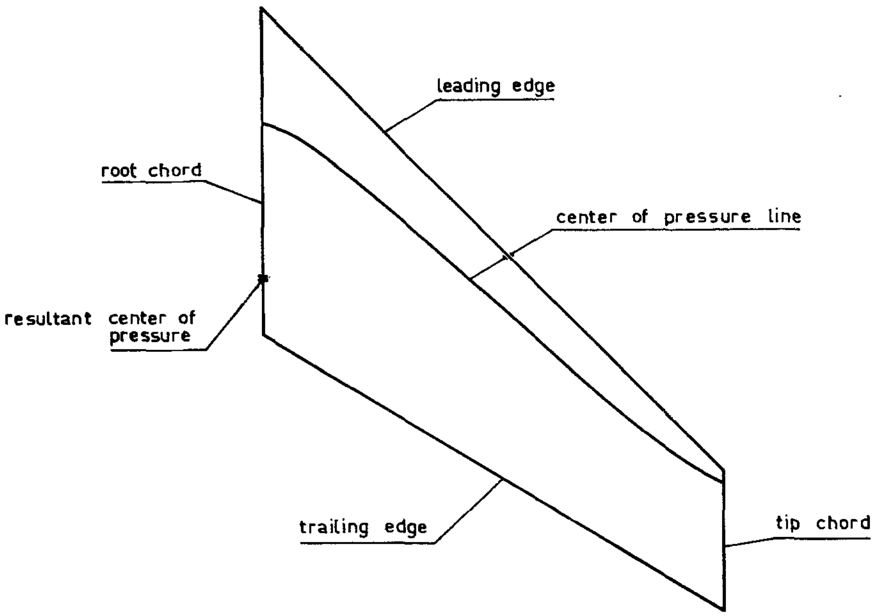

Fig. 1. Wing planform and positions of center of pressure.

The number of pivotal points in spanwise direction varied from m+1=8 to 24, while that in chordwise direction varied from R+1=1 to 5. These numbers are only limited by the number of fast memory cells available in the computer. The computer used was the Telefunken TR4 of the University of Groningen.

There is an Algol 60 program available, see [7], which calculates and prints the following quantities for a symmetrically loaded wing.

(i) the spanwise sections η_ν, $\nu=1,2,\ldots,m+1$.
(ii) the functions $a_r(\eta_\nu)$, $r=0,1,\ldots,R$; $\nu=1,2,\ldots,m+1$.
(iii) the spanwise lift distribution
(iv) the position of the sectional centre of pressure
(v) the lift coefficient of the whole wing
(vi) the position of the resultant centre of pressure
(vii) the spanwise distribution of the induced drag
(viii) the total induced drag obtained by integration of (vii)
(ix) the total induced drag obtained from the wake.

The spanwise lift distribution and the spanwise distribution of induced drag have been given in figs. 2 and 3, respectively. The positions of the sectional centers of pressure and of the resultant center of pressure have been added in fig. 1. Finally, fig. 4 shows the functions $a_r(\eta)$ for the case of m+1=20 spanwise points and R+1=4 chordwise points.

It is necessary that in the middle section of a swept wing $a_0(\eta)$ decreases to zero in order to keep the downwash finite at the foremost point of the wing. The pressure in the middle section then is due to the components $a_1(\eta)$, $a_2(\eta)$, etc. The quick change of these components is difficult to grasp unless m and R are taken rather large and this is one of the reasons for the slow convergence of the components $a_r(\eta)$ when m and R are increased. Fig. 4 is only valid for the number of pivotal points mentioned, but an analogous figure for other numbers of points would be only slightly different.

The decrease of $a_0(\eta)$ in the middle section of a swept wing implies the vanishing of the section force there. Hence, the sectional induced drag is large near the middle suction and even becomes negative for the outer sections of a swept wing (fig. 3).

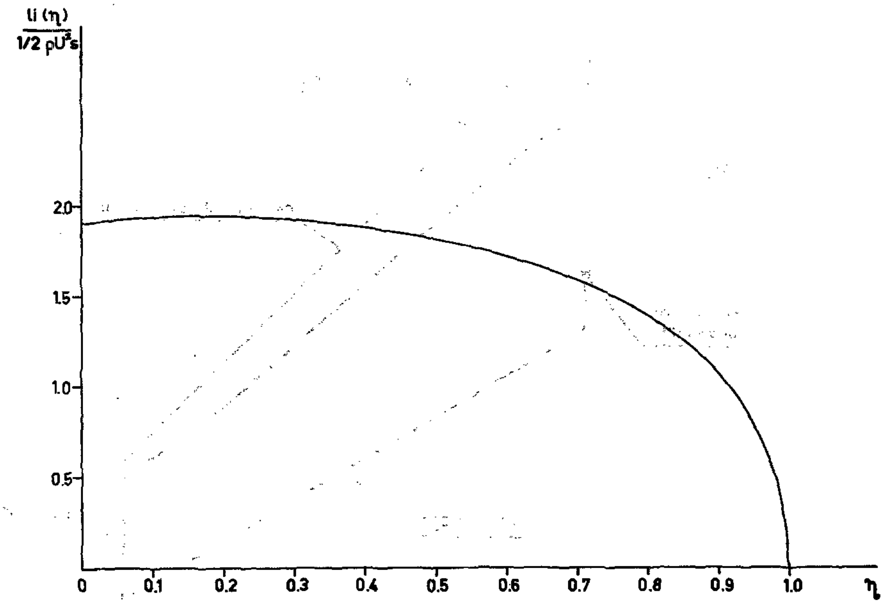

Fig. 2. Spanwise lift distribution.

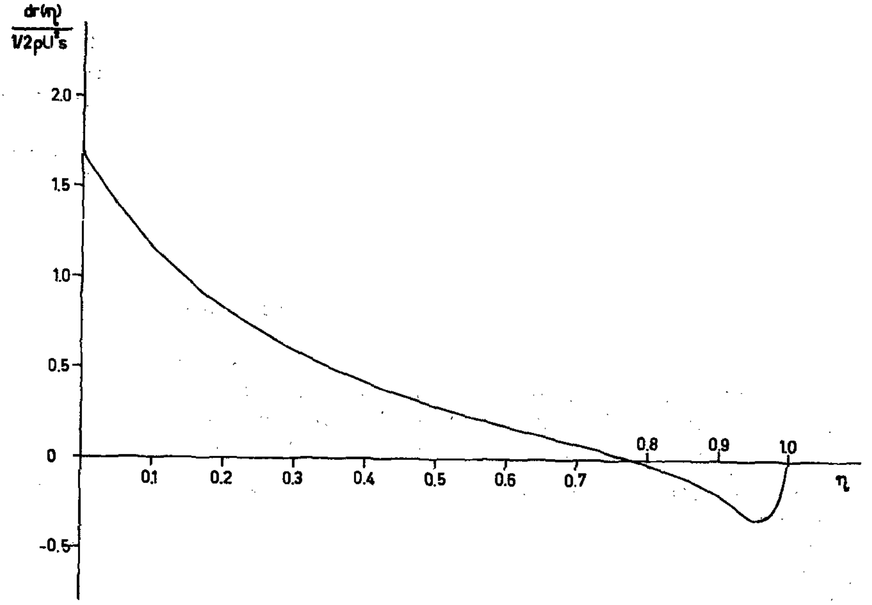

Fig. 3. Spanwise distribution of induced drag.

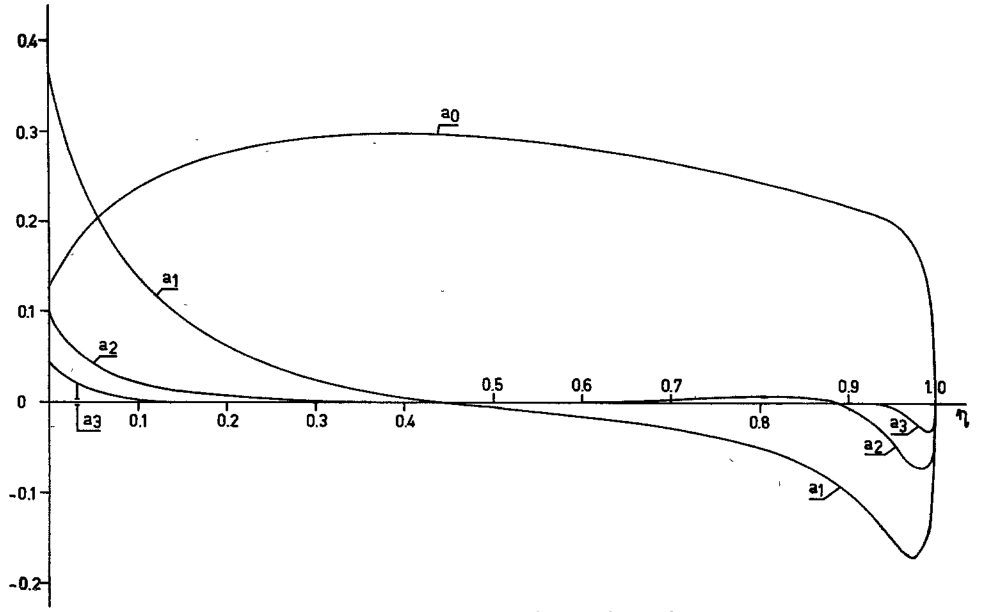

Fig. 4. The functions $a_r(\eta)$; m+1=20; R+1=4.

7. *References.*

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[Received November 30, 1966]